Ambiguity and Long-Run Cooperation in Strategic Games

Appendix

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A Capacities and neo-additive capacities

Given a finite space X and its correspondent power set 2^X, a capacity $v : 2^X \to \mathbb{R}_+$ is a function that satisfies

$$\begin{aligned} v(\phi) &= 0, \\ v(A) &\leq v(B) \quad \text{if } A \subseteq B, \\ v(X) &= 1. \end{aligned}$$

A capacity is said to be convex if $v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$ (concave if the relation holds with $\geq$). Hence, capacities do not necessarily comply the additivity law of probabilities. Integrating a function $f : X \to \mathbb{R}$ with respect to a capacity v (the analog of an expectation in the additive probability framework) is done by using Choquet integrals (Choquet, 1954). When the capacity is additive, the Choquet integral is equivalent to the Riemann integral. Capacities can

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capture ambiguous beliefs since, given their non-additivity, the sum of the likelihood assigned to the realization of the different states does not necessarily add to one.

A neo-additive capacity, proposed by Chateauneuf et al. (2007), is a particular type of capacity defined by

$$v(A) \quad := \quad (1-\delta)\pi(A)+\delta\mu_{1-\alpha}^{\mathscr{N}}(A),$$

for all $A \subset X$, where $\delta \in [0,1]$, π is an additive probability distribution defined over X, and $\mu_{1-\alpha}^{\mathscr{N}}$ is a Hurwicz capacity exactly congruent with $\mathscr{N} \subset X$ with an $1-\alpha \in [0,1]$ degree of optimism, defined by[1]

$$\mu_{1-\alpha}^{\mathscr{N}}(A) = \begin{cases} 0 & \text{if } A \in \mathscr{N}, \\ 1-\alpha & \text{if } A \notin \mathscr{N} \text{ and } S \setminus A \notin \mathscr{N}, \\ 1 & \text{if } S \setminus A \in \mathscr{N}, \end{cases}$$

where S is the set of all possible states and $\mathscr{N} \subset X$ is the set of null events, i.e., the set of states whose realization is impossible. Chateauneuf et al. (2007) show that the Choquet integral of a neo-additive capacity is given by (1) and that axiomatizes a utility function under ambiguity.

B Proof of Proposition II

Assume that $s_{-i,-1} = c$ for $i = 1,2$. Therefore, history allows asking about the conditions for sustaining the cooperative sequence as a dynamic equilibrium. We say $(\{s_{it} = c\})_{t=0}^{\infty}$ for $i = 1,2$ is an equilibrium if the present value of always cooperating is larger than or equal to the present value of deviating from the cooperative agreement and then being punished by the proposed scheme.

Always-cooperating strategy In $t = 0$ the expected payoff of the cooperative agreement is

$$v_c^* \quad := \quad \delta((1-\alpha)M(c)+\alpha m(c))+(1-\delta)u(c,c).$$

In $t = 1$, the individual sees what the counterpart played in $t = 0$. If the counterpart played c in the previous period, then the individual keeps playing c and makes no update on the parameters.

[1] Consequently, α denotes the degree of pessimism (ambiguity aversion).

Therefore, the expected payoff is again v_c^*. The individual predicts that this will happen with probability ϕ_c, which is bounded from below by $(1-\delta)$.[2] If the counterpart deviated in the previous period (i.e., played any action $s_{-i} \in S$ different from c), then the individual punishes the counterpart by playing n and updates the ambiguity parameter, δ, to 1. This situation gives an expected payoff of

$$v_n^u := (1-\alpha)M(n) + \alpha m(n).$$

Then, the expected payoff of the cooperative agreement in $t=1$, seen from $t=0$, is given by $\phi_c v_c^* + (1-\phi_c)v_n^u$. Note that once parameters are updated they are no longer revised, as the individual plays n forever and does not make any further updates, regardless of the other player's future actions.

A similar argument is applied recursively for future periods. If the counterpart played c in $t=0$ and, therefore, the individual plays c in $t=1$, in $t=2$ sees what the counterpart played in $t=1$ and decides how to behave following the rule described above. Hence, the expected payoff of the cooperative agreement in $t=2$, seen from $t=0$, is $\phi_c^2 v_c^* + \phi_c(1-\phi_c)v_n^u + (1-\phi_c)v_n^u = \phi_c^2 v_c^* + (1-\phi_c^2)v_n^u$. Similar calculations allow to conclude that the expected payoff of the cooperative equilibrium in $t=T$, seen from $t=0$, is $\phi_c^T v_c^* + (1-\phi_c^T)v_n^u$.

Thereby, given the subjective discount factor, β, the present value at $t=0$ of playing the cooperative strategy is

$$\begin{aligned} PV_c &= v_c^* + \beta\left[\phi_c v_c^* + (1-\phi_c)v_n^u\right] + \beta^2\left[\phi_c^2 v_c^* + (1-\phi_c^2)v_n^u\right] + \ldots \\ &= v_c^*\left[1 + \beta\phi_c + (\beta\phi_c)^2 + \ldots\right] + v_n^u\left[\beta(1-\phi_c) + \beta^2(1-\phi_c^2) + \ldots\right] \\ &= v_c^* \sum_{s\geq 0}(\beta\phi_c)^s + \beta v_n^u \sum_{s\geq 0}\beta^s - \beta\phi_c v_n^u \sum_{s\geq 0}(\beta\phi_c)^s \\ &= \frac{v_c^* - \beta\phi_c v_n^u}{1-\beta\phi_c} + \frac{\beta v_n^u}{1-\beta}. \end{aligned} \tag{B.1}$$

Deviating strategy In $t=0$, the expected payoff of deviating from the cooperative agreement is

$$v_d^* := \delta((1-\alpha)M(d) + \alpha m(d)) + (1-\delta)u(d,c).$$

[2] It could be higher, for example, if $M(c) = u(c,c)$.

After deviating, the individual knows that the counterpart will punish her by playing n forever, so she will play n forever as well. Nevertheless, other player's actions may induce updating on the individual's parameters if they do not match the expected behavior. The other player's expected behavior in $t = 0$ is to play c (which the individual predicts will happen with probability ϕ_c), and in $t \geq 1$ is to play n (which the individual predicts will happen with probability ϕ_n, which is also bounded from below by $(1-\delta)$).[3]

In $t = 1$ the individual sees what the counterpart played in $t = 0$. If the counterpart played c in the previous period, then the individual makes no update and perceives an expected payoff of

$$v_n^* \;:=\; \delta((1-\alpha)M(n) + \alpha m(n)) + (1-\delta)u(n,n).$$

If the counterpart played any action $s_{-i} \in S$ different from c in the previous period, then the individual updates the ambiguity parameter to 1 and perceives an expected payoff of v_n^u. Again, in the latter situation, parameters will not revised. Adding up, the expected payoff of the deviating strategy in $t = 1$, seen from $t = 0$, is $\phi_c v_n^* + (1-\phi_c)v_n^u$.

Again, a recursive argument is followed. If the counterpart played c in $t = 0$ and, therefore, the individual made no update in $t = 1$, in $t = 2$ the player sees what the counterpart played in $t = 1$ and decides how to behave. If the counterpart played n, then the individual makes no update and perceives an expected payoff of v_n^*. Conversely, if the counterpart deviated from the expected behavior, then the expected payoff is v_n^u. Hence, the expected payoff of the deviating strategy in $t = 2$, seen from $t = 0$, is $\phi_c\phi_n v_n^* + \phi_c(1-\phi_n)v_n^u + (1-\phi_c)v_n^u = \phi_c\phi_n v_n^* + (1-\phi_c\phi_n)v_n^u$. Similar calculations allow to conclude that the expected payoff of the deviating strategy in $t = T$, seen from $t = 0$, is $\phi_c\phi_n^{(T-1)}v_n^* + \left(1-\phi_c\phi_n^{(T-1)}\right)v_n^u$.

Thereby, given the subjective discount factor, β, the present value of playing the deviating

[3] It could be higher, for example, if $m(n) = u(n,n)$.

strategy is

$$
\begin{aligned}
PV_d &= v_d^* + \beta\left[\phi_c v_n^* + (1-\phi_c)v_n^u\right] + \beta^2\left[\phi_c\phi_n v_n^* + (1-\phi_c\phi_n)v_n^u\right] + \ldots \\
&= v_d^* + \beta\phi_c v_n^*\left[1 + \beta\phi_n + (\beta\phi_n)^2 + \ldots\right] + \beta v_n^u\left[1 + \beta + \beta^2 + \ldots\right] - \beta\phi_c v_n^u\left[1 + \beta\phi_n + (\beta\phi_n)^2 + \ldots\right] \\
&= v_d^* + \beta\phi_c v_n^* \sum_{s\geq 0}(\beta\phi_n)^s + \beta v_n^u \sum_{s\geq 0}\beta^s - \beta\phi_c v_n^u \sum_{s\geq 0}(\beta\phi_n)^s \\
&= v_d^* + \frac{\beta\phi_c(v_n^* - v_n^u)}{1-\beta\phi_n} + \frac{\beta v_n^u}{1-\beta}. \qquad \text{(B.2)}
\end{aligned}
$$

Putting together (B.1) and (B.2) yields (7). ■

C Comparative statics in the repeated PD with $\alpha = 0.5$

In the PD

$$
\begin{aligned}
v_c^* &= \delta((1-\alpha)R + \alpha Q) + (1-\delta)R, \qquad \text{(C.1)} \\
v_n^* &= \delta((1-\alpha)T + \alpha P) + (1-\delta)P, \qquad \text{(C.2)} \\
v_d^* &= \delta((1-\alpha)T + \alpha P) + (1-\delta)T. \qquad \text{(C.3)}
\end{aligned}
$$

With $\alpha = 0.5$, $\phi_n = \phi_c = \phi = 1 - \delta/2$. When $\phi_c = \phi_n = \phi$, (7) is reduced to

$$
\frac{v_c^* - \beta\phi v_n^*}{1-\beta\phi} \geq v_d^*,
$$

which implies that,

$$
\beta^* = \phi^{-1}\frac{(v_d^* - v_c^*)}{(v_d^* - v_n^*)}, \qquad \text{(C.4)}
$$

Using (C.1), (C.2), (C.3), and (C.4), we have that

$$
\beta^* = \frac{\phi^2(T-R) + \frac{\phi\delta}{2}(P-Q)}{(1-\delta)(T-P)}.
$$

The objective is to determine the sign of $\frac{\partial\beta^*}{\partial\delta}$. This is the derivative of a ratio, so the sign is determined by the sign of the numerator of the derivative, which is

$$\left[2\phi\frac{\partial\phi}{\partial\delta}(T-R)+\frac{(P-Q)}{2}\left(\phi+\frac{\partial\phi}{\partial\delta}\delta\right)\right](1-\delta)(T-P)+\left[\phi^2(T-R)+\frac{\phi\delta}{2}(P-Q)\right](T-P).$$

The first bracket is the only expression with undefined sign (all the rest are positive). Then, if it is positive, then $\frac{\partial\beta^*}{\partial\delta}>0$. Since $\frac{\partial\phi}{\partial\delta}=-\frac{1}{2}$, the first bracket is equivalent to

$$P\frac{(1-\delta)}{2}-Q\frac{(1-\delta)}{2}-T\left(1-\frac{\delta}{2}\right)+R\left(1-\frac{\delta}{2}\right).$$

Adding and subtracting $\frac{Q\delta}{4}$, the previous expression can be written as

$$P\frac{(1-\delta)}{2}+\left(R-\frac{(Q+T)}{2}\right)\left(1-\frac{\delta}{2}\right)+\frac{T}{2}\left(1-\frac{\delta}{2}\right)+\frac{Q\delta}{4},$$

which is positive since $R>\frac{Q+T}{2}$. Then, we conclude that $\frac{\partial\beta^*}{\partial\delta}>0$.

D Duopoly models: Derivations

D.1 Cournot model

The objective function –equation (9)– has discontinuities since the price (i) depends on the firm's production, and (ii) is bounded from below by zero. In these cases, the strategy is to solve the problem in the different scenarios (including the kinks), checking that the solutions lie within the relevant strategy space, and then, for each parametrization, choose the feasible solution that maximizes the expected profits.

Case 1 Assume that the price is positive in both cases. Taking the first order condition leads to the following reaction function

$$q(q_j)=\frac{\delta(1-\alpha)A+(1-\delta)(A-bq_j)-k}{2b(1-\delta\alpha)}.$$

The fixed point of this reaction function is $q = \frac{\delta(1-\alpha)A+(1-\delta)A-k}{2b(1-\delta\alpha)+b(1-\delta)}$. This optimization assumes that $\max\left\{A-b(q+q_j),0\right\} = A-b(q+q_j)$. Then, solutions are feasible if (i) when computing q^n using the non-symmetric assumption, $q^n \leq \frac{2A+k}{3b}$, (ii) when computing q^n using the symmetric assumption, $q^n \leq \frac{A}{2b}$, and (iii) when computing q^d, $q^d \leq \frac{3A+k}{4b}$.

Case 2 When q is above the thresholds, the objective function is $\delta(1-\alpha)\max\{A-bq,0\}\,q-kq$. Regardless of the situation, the solution is $q^* = \frac{\delta(1-\alpha)A-k}{2b\delta(1-\alpha)}$ which is always smaller than $\frac{A}{b}$ and, therefore, the second kink is redundant.

D.2 Bertrand model

The objective function –equation (10)– has discontinuities since the quantity (i) depends on the firm's price, and (ii) is bounded from below by zero. In these cases, the strategy is to solve the problem in the different scenarios (including the kinks), checking that the solutions lie within the relevant strategy space, and then, for each parametrization, choose the feasible solution that maximizes the expected profits.

Case 1 Assume that the quantity is positive in all cases. Taking the first order condition leads to the following reaction function

$$p(p_j) = \frac{a+b_1\left[(1-\delta)p_j+\delta(1-\alpha)K+\delta\alpha k\right]+b_2k}{2b_2}.$$

The fixed point of this reaction function is $p = \frac{a+b_1(\delta(1-\alpha)K+\delta\alpha k)+b_2k}{2b_2-b_1(1-\delta)}$. This optimization assumes that $\max\{a+b_1k-b_2p,0\} = a+b_1k-b_2p$. Then, solutions are feasible if $p \leq \frac{a+b_1k}{b_2}$. Note that under our simplifying assumption that $K = a/b_2$, this threshold is never met.

Case 2 When p is above the threshold, the objective function is $(p-k)\left(\delta(1-\alpha)\max\{a+b_1K-b_2p,0\}\right.$ $\left.+(1-\delta)\max\left\{a+b_1p_j-b_2p,0\right\}\right)$. Assume that the quantity is positive in both cases. Taking the first order condition leads to the following reaction function

$$p(p_j) = \frac{a}{2b_2}+\frac{b_1\left(\delta(1-\alpha)K+(1-\delta)p_j\right)}{2b_2(1-\delta\alpha)}+\frac{k}{2}.$$

The fixed point of this reaction function is $p = \frac{(a+b_2k)(1-\delta\alpha)+b_1\delta(1-\alpha)K}{2b_2(1-\delta\alpha)-b_1(1-\delta)}$. This optimization assumes that $\max\left\{a+b_1p_j-b_2p,0\right\} = a+b_1p_j-b_2p$. Then, solutions are feasible if (i) when computing p^n using the non-symmetric assumption, $\frac{a+b_1k}{b_2} \le p^n \le \frac{1}{b_2}\left(a+\frac{b_1(a+b_2k)}{2b_2-b_1}\right)$, (ii) when computing p^n using the symmetric assumption, $\frac{a+b_1k}{b_2} \le p^n \le \frac{a}{b_2-b_1}$, and (iii) when computing p^d, $\frac{a+b_1k}{b_2} \le p^d \le \frac{1}{b_2}\left(a+\frac{b_1}{2}\left(\frac{a}{b_2-b_1}+k\right)\right)$.

Case 3 When p is above the thresholds, the objective function is $(p-k)\delta(1-\alpha)\max\{a+b_1K-b_2p,0\}$. Assume the quantity is positive. Then, the solution is $p^* = \frac{a+b_1K+b_2k}{2b_2}$. In this case, quantity is positive whenever $a+b_1K-b_2p>0$. Then, $p = \frac{a+b_1K}{b_2}$ defines an additional kink. For any $p \ge \frac{a+b_1K}{b_2}$, the expected profit is zero.

D.3 Solution

To numerically solve these maximization problems, for every (δ,α) combination, we (i) compute the optimal quantity/price in the different cases, (ii) check that the optimal quantity/price is consistent with the thresholds, (iii) compute the expected profits for all feasible cases (including kinks), and (iv) select the feasible case that gives the highest profit.

References

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